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Study of a Fractional Integral Formula Involving Fractional Exponential Function

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we obtain a fractional integral formula involving fractional exponential function. Moreover, we provide some examples to illustrate our result. In fact, our formula is a generalization of traditional calculus formula.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral formula, fractional exponential function.**

I. INTRODUCTION

Fractional calculus is a mathematical analysis tool used to study arbitrary order derivatives and integrals. It unifies and extends the concepts of integer order derivatives and integrals. Generally, many scientists do not know these fractional integrals and derivatives, and they have not been used in pure mathematical context until recent years. However, in the past few decades, the fractional integrals and derivatives have frequently appeared in many scientific fields such as mechanics, viscoelasticity, physics, economics and engineering [1-8].

Until now, the definition of fractional derivative is not unique. The commonly used definitions are Riemann-Liouvellie (R-L) fractional derivative, Caputo definition of fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [9-13]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we obtain a fractional integral formula. In addition, we give some examples to illustrate our result. In fact, our formula is a generalization of classical calculus formula.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([14])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

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Proposition 2.2 ([15])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x - x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x - x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Definition 2.3 ([16])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([17])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x - x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([18])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{9}$$

Theorem 2.6 (integration by parts for fractional calculus) ([19])**:** *Suppose that* $0 < \alpha \leq 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions on an interval* I *containing* $[a, b]$, *then*

$$\left({}_aI_b^{\alpha}\right)\left[f_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^{\alpha}\right)[g_\alpha(x^\alpha)]\right] = \left[f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^{\alpha}\right)\left[g_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^{\alpha}\right)[f_\alpha(x^\alpha)]\right]. \tag{10}$$

III. MAIN RESULT AND EXAMPLES

In this section, we obtain a fractional integral formula involving fractional exponential function. On the other hand, some examples are provided to illustrate our result.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *and* p, q, r *be real numbers, then the* α*-fractional integral*

$$\left({}_0I_x^{\alpha}\right)\left[\left(rp\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha r} + rq\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (r-1)} + p\right) \otimes_\alpha E_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha r}\right)\right]$$

$$=\left(p\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+q\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)-q. \tag{11}$$

Proof $\left({}_{0}I_{x}^{\alpha}\right)\left[\left(rp\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}+rq\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(r-1)}+p\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[rp\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]+\left({}_{0}I_{x}^{\alpha}\right)\left[rq\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(r-1)}\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]$$

$$+\left({}_{0}I_{x}^{\alpha}\right)\left[p\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]$$

$$=p\cdot\left({}_{0}I_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left({}_{0}D_{x}^{\alpha}\right)\left[E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]\right]+q\cdot\left({}_{0}I_{x}^{\alpha}\right)\left[\left({}_{0}D_{x}^{\alpha}\right)\left[E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]\right]$$

$$+p\cdot\left({}_{0}I_{x}^{\alpha}\right)\left[E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]$$

$$=p\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)-p\cdot\left({}_{0}I_{x}^{\alpha}\right)\left[E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]+q\cdot E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)-q$$

$$+p\cdot\left({}_{0}I_{x}^{\alpha}\right)\left[E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)\right]\ \text{(by integration by parts for fractional calculus)}$$

$$=\left(p\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+q\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}\right)-q\ .\qquad \text{Q.e.d.}$$

Example 3.2: If $0<\alpha\leq 1$, then by Theorem 3.1 we have

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}+1\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right)\right]=\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}\right). \tag{12}$$

And

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(6\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}3}+3\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}+2\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}3}\right)\right]$$

$$=\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)\otimes_{\alpha}E_{\alpha}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}3}\right)-1\ . \tag{13}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study a fractional integral formula involving fractional exponential function. In addition, we provide some examples to illustrate our result. In fact, our formula is a generalization of ordinary calculus formula. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in fractional differential equations and engineering mathematics.

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